

# On uniform flag bundles on Fano manifolds

Roberto Muñoz, Gianluca Occhetta, and Luis E. Solá Conde

**ABSTRACT.** As a natural extension of the theory of uniform vector bundles on Fano manifolds, we consider the case of uniform principal bundles, proposing the associated flag bundles as a natural geometric framework to study such bundles. In this paper we introduce the basic definitions and results regarding those bundles, focusing on the problem of their reducibility and diagonalizability in low rank.

## 1. Introduction

Although the fact that a vector bundle over the complex projective line  $\mathbb{P}^1$  splits as a direct sum of line bundles is a theorem whose history goes back to the end of the nineteenth century, it was not until the 1950's that it achieved its modern form as posed by A. Grothendieck. Working upon the ideas developed previously by the french school of Cartan and Borel, he considered vector bundles as geometric realizations of principal  $G$ -bundles, with  $G$  reductive, via certain representations of the group  $G$ , and showed that every principal bundle over  $\mathbb{P}^1$  is determined uniquely by a co-character of a Cartan subgroup of  $G$ . It is then this co-character the invariant that determines the splitting type of any vector bundle associated to the principal bundle via a given representation of  $G$ .

For varieties different from  $\mathbb{P}^1$  the situation is far more complicated, since even the simplest varieties may admit non isomorphic vector bundles with the same Chern classes. In the particular case in which the base variety contains rational curves, for instance for Fano manifolds, a vector bundle has a splitting type on each of these curves and the study of these splitting types may already help to understand some of the properties of the bundle.

This idea is particularly useful when considering vector bundles that are uniform with respect to an unsplit dominating family of rational curves. Remarkably, uniform bundles of rank small enough are direct sums of line bundles over varieties such as the projective space ([23, 22]), quadrics ([2]) and some other rational homogeneous spaces ([9, 15]). In many of these cases, the bound on the rank for which this result hold (that depends on the base variety) has been proved to be sharp. Furthermore, over some of these varieties low rank indecomposable vector

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2010 *Mathematics Subject Classification.* Primary 14J45; Secondary 14E30, 14M15, 14M17.

First and third author partially supported by the Spanish government project MTM2015-65968-P. Second author supported by PRIN project "Geometria delle varietà algebriche". Second and third author supported by the Department of Mathematics of the University of Trento.

bundles are homogeneous ([3, 6, 5, 24, 15, 14]), which in turn suggests that one should not only look at the vector bundles that one wants to study, but also at the subjacent principal bundles, and their relation with the group of automorphisms of the base variety.

On the other hand, within the framework of the theory of Fano manifolds one often finds the need of studying rational homogeneous bundles; the first nontrivial examples appear in relation with the VMRT of a rational homogeneous space (see [11, 13]). Although one may always embed a rational homogeneous bundle in a projective bundle, whose geometry is well known, the general facts on rational homogeneous spaces and principal bundles should allow a direct study of the geometry of such bundles, or of any other convenient geometric incarnations of the subjacent principal bundle. In this sense, a particularly natural choice, is the associated flag bundle, which is constructed upon the action of the defining group  $G$  on the flag manifold  $G/B$ . A geometrical reason for this choice is that flag manifolds are particularly simple when one looks at their families of minimal rational curves; in fact, the intersection properties of these families contain the necessary information to reconstruct the action of the group  $G$  (see [18, 19]).

This paper is the first of a project in which we study uniform principal  $G$ -bundles ( $G$  semisimple) over Fano manifolds, with special interest on decomposability questions about them. Our goal is to develop the background necessary to address these questions, and to make it accessible to the experts on vector bundles on Fano manifolds.

The structure of the paper is the following: we start in Section 2 with some generalities on flag bundles and their relation to principal  $G$ -bundles. We pay special attention to certain filtrations of their relative tangent bundles, that will be useful later on. In section 3 we define decomposability, reducibility and diagonalizability for flag bundles, generalizing the different standard decomposability notions of vector bundles. We also discuss the interactions among these concepts, relating them with the existence of sections of the associated rational homogeneous bundles (see Proposition 3.6, and Section 5.1).

The notion of uniform flag bundle is settled in Section 4 by means of the concept of tag of a principal  $G$ -bundle on a rational curve. The tag is defined in [19] as a  $\mathrm{rk}(G)$ -vector of non-negative integers and it is a geometric counterpart of Grothendieck's classifying co-character. We prove in this section a characterization of trivial flag bundles in terms of their tags with respect to certain families of rational curves. Our proof is complementary to the different proofs characterizing trivial vector bundles in terms of their restrictions to curves ([1, 4, 21], see also [16, Proposition 2.4]), stressing the interplay between both approaches –flag bundle versus vector bundle– to these questions.

The last section is devoted to the study of criteria of reducibility and diagonalizability for uniform flag bundles. In this case it is convenient to consider a special type of reducibility, named *uniform reducibility*, defined in Section 5.1 upon the particular family of rational curves with respect to which the uniformity is defined. For instance, in Lemma 5.3 we show that for a uniformly reducible flag bundle, its diagonalizability reduces to the diagonalizability of an auxiliary flag bundle of lower rank. Roughly speaking, one could say that uniform flag bundles whose tag is sufficiently positive are reducible. In this spirit, we show in Proposition 5.14 that, under certain technical conditions on the family of rational curves (see Setup

5.6), the existence of elements bigger than one in the tag implies the reducibility of the bundle. In particular, under those conditions, the problem of diagonalizability of uniform flag bundles of low rank reduces to that of flag bundles tagged with zeros and ones. Finally we prove (see Theorem 5.15 for a precise statement) that not every tag with ones and zeros may occur on a non diagonalizable bundle, by showing that every 1 in the tag must be conveniently isolated by zeros.

## 2. Setup and preliminaries

Along this paper  $X$  will denote a complex projective algebraic variety. A  $Z$ -bundle over  $X$  is a smooth morphism  $\pi : Y \rightarrow X$  whose scheme theoretical fibers are isomorphic to  $Z$ . We will be mostly interested in the case in which  $Z$  is of the form  $G/B$  or  $G/P$ , where  $G$  is a semisimple complex algebraic group, with Dynkin diagram  $\mathcal{D}$ ,  $B \subset G$  is a Borel subgroup, and  $P \subset G$  is a parabolic subgroup. A  $G/B$ -bundle on  $X$  is then, by definition, a smooth morphism  $\pi : Y \rightarrow X$  whose fibers are isomorphic to  $G/B$ . Note that, given  $\pi$ , we may choose (see [19, Remark 2.1]),  $G$  to be the identity component of the automorphism group of  $G/B$ , so that  $\pi$  is determined by a cocycle  $\theta \in H^1(X, G)$  (by abuse of notation, we mean here the cohomology of the sheafified group  $G$  on the analytic space associated to  $X$ ). Moreover,  $\pi : Y \rightarrow X$  may be obtained as a quotient over  $X$  of the  $G$ -principal bundle  $\pi_G : E \rightarrow X$  associated to  $\theta$  (in fact,  $E$  would be a  $B$ -principal bundle over  $Y$ ); alternatively, we may identify  $Y$  with the algebraic variety

$$E \times^G G/B := (E \times G/B) / \sim, \quad (e, gB) \sim (eh, h^{-1}gB), \quad \forall h \in G,$$

and then  $\pi$  corresponds to the natural map sending the class of  $(e, gB)$  to  $\pi_G(e)$ .

If we consider a maximal torus  $H \subset B$ , it determines a root system  $\Phi$ , whose Weyl group  $W$  is isomorphic to the quotient  $N(H)/H$  of the normalizer  $N(H)$  of  $H$  in  $G$ . Within  $\Phi$ ,  $B$  provides a base of positive simple roots  $\Delta$ . Finally, as usual, we consider the Dynkin diagram  $\mathcal{D}$  associated with  $\Phi$ . We will always choose an ordering of the set of simple roots  $\Delta = \{\alpha_1, \dots, \alpha_n\}$ , (in the case of simple algebraic group we will always choose the ordering of [10, p. 58]), and denote by  $D$  the set of indices  $\{1, \dots, n\}$ . By definition, the *rank* of the semisimple group  $G$  is defined as  $\text{rk}(G) := \dim H = \sharp(\Delta) = n$ .

We denote by  $r_i$  the reflection associated to  $\alpha_i$ . Then, for every subset  $I \subset D$  we may consider a parabolic subgroup  $P(I)$  defined by  $P(I) := BW_I B$ , where  $W_I \subset W$  is the subgroup of  $W$  generated by the reflections  $r_i$ ,  $i \in I$ . Going back to our setting, for every such subset  $I \subset D$  there is a factorization:

$$(1) \quad \begin{array}{ccccc} & & \pi & & \\ & \nearrow & & \searrow & \\ Y & \xrightarrow{\rho_I} & Y_I & \xrightarrow{\pi_I} & X \end{array}$$

where  $Y_I := E \times^G G/P(I)$ . In the case in which  $I = \{i\}$ , we will simply write  $\rho_i, \pi_i, Y_i$ .

Finally, we denote by  $N^1(Y|X)$  the cokernel of the pull-back map  $N^1(X) \rightarrow N^1(Y)$ , between the real vector spaces of classes of  $\mathbb{R}$ -divisors in  $X$  and  $Y$ . It is a vector space of dimension equal to the Picard number of  $G/B$  (that we denote by  $n$ ), that we may (and will) identify with the linear subspace of  $N^1(Y)$  generated by the linearly independent set  $\{-K_i, i \in D\}$ , where  $-K_i$  denotes the relative anticanonical divisor of the elementary contraction  $\rho_i : Y \rightarrow Y_i$ , for any  $i \in D$ . The numerical class of a fiber of this contraction is denoted by  $\Gamma_i$ .

**2.1. Standard constructions.** We include here some classical constructions with principal and fiber bundles.

1. *Pullback.* Given a  $Z$ -bundle  $\pi : Y \rightarrow X$ , and a morphism  $f : X' \rightarrow X$ , the fiber product  $Y \times_X X'$  has a natural structure of  $Z$ -bundle over  $X'$ . In the case in which  $Z = G/B$  and  $\pi$  is determined by a cocycle  $\theta \in H^1(X, G)$ , the bundle  $Y \times_X X' \rightarrow X'$  corresponds to the image of  $\theta$  by the pullback map  $H^1(X, G) \rightarrow H^1(X', G)$ .

2. *Extension.* Given any morphism of Lie groups  $G \rightarrow G'$  ( $G'$  semisimple), there is a natural map  $H^1(X, G) \rightarrow H^1(X, G')$  that sends  $\theta$  to a cocycle  $\theta'$  defining a  $G'/B'$ -bundle, which may also be described as the  $G'/B'$ -bundle  $E \times^G G'/B'$ , where we consider the action of  $G$  on  $G'/B'$  induced by the map  $G \rightarrow G'$ .

3. *Reduction.* Conversely, if  $\theta \in H^1(X, G)$  (defining a  $G/B$ -bundle  $\pi : Y \rightarrow X$ ), and  $f : G' \rightarrow G$  is a homomorphism of algebraic groups, we say that  $\theta$  admits a *reduction to  $G'$*  if  $\theta$  lies in the image of the natural map  $H^1(X, G') \rightarrow H^1(X, G)$ . In the case in which the map  $f$  is the inclusion  $P \hookrightarrow G$ , the reduction to  $P$  is equivalent to the existence of a section  $s$  of  $\pi_I : Y_I \rightarrow X$  (where  $I$  is the set of nodes of  $D$  defining  $P$ ). In particular, in the case  $P = B$ , the reduction of  $\theta$  to  $B$  is equivalent to the existence of a section of  $\pi$ . Moreover, considering the *semisimple part*  $G_P$  of  $P$  (which is, by definition, the quotient of  $P$  by its unipotent subgroup, and then by the center of the image), and its Borel subgroup  $B_P \subset G_P$ , the extension of  $\theta$  to  $G_P$  defines a  $G_P/B_P$ -bundle  $\pi^P : Y^P \rightarrow X$ . Furthermore, by construction,  $Y^P$  admits an embedding  $i_P$  into  $Y$  satisfying that  $s \circ \pi^P = \pi_I \circ i_P$ .

4. *Product.* Given two semisimple groups,  $G, G'$ , and two flag bundles  $\pi : Y \rightarrow X$ ,  $\pi' : Y' \rightarrow X$ , determined by cocycles  $\theta \in H^1(X, G)$  and  $\theta' \in H^1(X, G')$ , and given any morphism  $\rho : G \times G' \rightarrow G''$ , the cocycle  $\rho(\theta, \theta') \in H^1(X, G'')$  defines a flag bundle over  $X$ . Even in the case in which  $\rho$  is injective, the flag bundle obtained is not, in general, the fiber product  $Y \times_X Y'$ .

**2.2. Filtrations of the relative tangent bundle.** Let  $m$  denote the dimension of  $G/B$ , which equals the cardinality of  $\Phi^+ \subset \Phi$ , which is defined as the set of roots that are nonnegative linear combinations of elements of the base  $\Delta$ . A total ordering  $(L_1, L_2, \dots, L_m)$  of the elements of  $\Phi^+$  is called *admissible* if, for every  $L_j, L_{j'}, L_{j''} \in \Phi^+$ ,  $L_j + L_{j'} = L_{j''}$  implies that  $j, j' < j''$ . Note that, for instance, any total ordering of  $\Phi^+$  satisfying  $\text{ht}(L_j) \leq \text{ht}(L_{j+1})$  (where the *height* of a positive root is defined as  $\text{ht}(L) := \sum_i a_i$  for  $L = -\sum_i a_i K_i$ ) is obviously admissible.

Then, following [17], for every admissible order we may construct a filtration of the relative tangent bundle  $T_{Y|X}$ :

$$0 = \mathcal{E}_m \subset \mathcal{E}_{m-1} \subset \dots \subset \mathcal{E}_0 = T_{Y|X},$$

whose quotients satisfy:

$$\mathcal{E}_r / \mathcal{E}_{r+1} \cong \mathcal{O}_Y(L_{m-r}), \quad \text{for } r \in \{0, \dots, m-1\}.$$

In particular we may state the following (see [19, Lemma 2.2] for an explicit formula):

LEMMA 2.1. *With the same notation as above, the relative anticanonical bundle  $-K_\pi$  is a positive integral combination of the relative anticanonical divisors  $-K_i$  of the elementary contractions  $\rho_i : Y \rightarrow Y_i$ .*

Given any set  $J \subset D$ , we denote by  $\Phi^+(J)$  the subset of  $\Phi^+$  consisting of positive roots that are linear combinations of the  $-K_i$ 's,  $i \in J$ .

DEFINITION 2.2. With the same notation as above, given a chain of subsets of  $D$ ,  $\emptyset = J_0 \subsetneq J_1 \subsetneq J_2 \subsetneq \cdots \subsetneq J_{k+1} = D$  an admissible ordering  $(L_1, \dots, L_m)$  of  $\Phi$  is said to be *compatible with*  $J_1 \subsetneq J_2 \subsetneq \cdots \subsetneq J_k$  if for every  $r = 1, \dots, k$  we have

$$\Phi^+(J_r) = \{L_1, L_2, \dots, L_{\sharp(\Phi^+(J_r))}\}.$$

REMARK 2.3. Given a chain of subsets of  $D$  as above, we may always find an admissible ordering of  $\Phi$  compatible with them. In fact it is enough to consider any total ordering such that the first  $\sharp(\Phi^+(J_r))$  positive roots belong to  $\Phi^+(J_r)$ , for every  $r$ , and such that the order of the elements  $L_{\sharp(\Phi^+(J_r))+1}, \dots, L_{\sharp(\Phi^+(J_{r+1}))}$  respects their height, for every  $r$ . Considering now the corresponding filtration of  $T_{Y|X}$  associated to such an ordering, we may write

$$\mathcal{E}_{m-\sharp(\Phi^+(J_r))} = T_{Y|Y_{J_r}}, \text{ for all } r.$$

In particular, quotienting every element of such a filtration by  $T_{Y|Y_{J_r}}$  we obtain a filtration of  $\rho_{J_r}^* T_{Y_{J_r}|X}$ :

$$0 = \overline{\mathcal{E}}_{m-\sharp(\Phi^+(J_r))} \subset \overline{\mathcal{E}}_{m-1} \subset \cdots \subset \overline{\mathcal{E}}_0 = \rho_{J_r}^* T_{Y_{J_r}|X},$$

with the same quotients:  $\overline{\mathcal{E}}_r / \overline{\mathcal{E}}_{r+1} \simeq \mathcal{E}_r / \mathcal{E}_{r+1} \cong \mathcal{O}_Y(L_{m-r})$ , for all  $r$ .

### 3. Reducibility, decomposability, and diagonalizability

A vector bundle is called decomposable if it is a direct sum of proper vector subbundles, and this can be seen at the level of the cocycle defining it. In fact, for a vector bundle on any variety  $X$  one may also consider the associated projective bundle, and its corresponding flag bundle  $\pi : Y \rightarrow X$ . If the vector bundle is decomposable, there exists a section of one of the corresponding Grassmannian bundles, associating to each  $x \in X$  the point corresponding to one of the summands of the bundle. The existence of this section is reflected in the fact that the bundle can be defined by using block-triangular matrices, but decomposability tells us also that we have a choice of a complementary subspace at every point, so that the bundle can be defined by using block-diagonal matrix. Following this idea, we will introduce in this section a notion of decomposability for flag bundles. Let us start with the following definitions.

DEFINITION 3.1. Let  $X$  be an algebraic variety,  $\pi : Y \rightarrow X$  be a  $G/B$ -bundle over  $X$  defined by a cocycle  $\theta \in H^1(X, G)$ , and  $I$  be a proper subset of  $D$ . Then the corresponding bundle  $\pi_I : Y_I \rightarrow X$  admits a section  $s_I : X \rightarrow Y_I$  if and only if the cocycle  $\theta$  lies in the image of the natural map  $H^1(X, P(I)) \rightarrow H^1(X, G)$ . In this case, we say that  $Y$  is *reducible with respect to*  $I$ .

REMARK 3.2. In particular, since for every  $I \subsetneq D$  the fiber product  $Y_I \times_X Y_I$  admits a section (the diagonal) over  $Y_I$ , it follows that the pull-back  $\pi_I^* \theta$  belongs to the image of the map  $H^1(Y_I, P(I)) \rightarrow H^1(Y_I, G)$ , for every  $I \subsetneq D$ , so that we may say that the pull-back bundle  $Y_I \times_X Y \rightarrow Y_I$  is reducible with respect to  $I$ , for every  $I$ . For  $I = \emptyset$  this is the analogue of the standard Splitting Principle for vector bundles, cf. [7, Section 3.2].

DEFINITION 3.3. We say that a  $G/B$ -bundle  $\pi : Y \rightarrow X$  is *decomposable* if there exists a proper subset  $I \subsetneq D$  such that:

- $Y$  is reducible with respect to  $I$ ,

- the cocycle  $\theta$  defining  $\pi$ , considered as an element of  $H^1(X, P(I))$  belongs to the image of the natural map  $H^1(X, L(I)) \rightarrow H^1(X, P(I))$ , where  $L(I)$  is a Levi part of  $P(I)$ .

Note that this map is an inclusion, since its composition with the natural map  $H^1(X, P(I)) \rightarrow H^1(X, L(I))$  is the identity.

DEFINITION 3.4. If the subset  $I$  defining the decomposability of  $\pi : Y \rightarrow X$  is empty, we say that  $\pi$  is *diagonalizable*. The reason for this name is that the Levi parts of  $B = P(\emptyset)$  are the Cartan subgroups of  $G$  contained in  $B$ , hence the definition is saying that  $\pi$  is defined by a cocycle in  $H^1(X, (\mathbb{C}^*)^n)$ . In particular, every vector bundle over  $X$  defined by this cocycle and a given linear representation of  $G$ , will be a direct sum of line bundles.

REMARK 3.5. In the case in which the Dynkin diagram of the group  $G$  is disconnected (that is, if  $G$  is semisimple, but not simple), it follows that the general fiber  $G/B$  is isomorphic to a product of flag varieties  $G_1/B_1 \times G_k/B_k$ , where every  $G_i$  is a simple algebraic group. If moreover  $X$  is simply connected, then the above decomposition holds globally, and  $Y$  is a fiber product of flag bundles  $Y_1 \times_X \dots \times_X Y_k$  over  $X$ . In this case  $Y$  is diagonalizable if and only if every  $Y_i$  is diagonalizable.

Note that any  $G/B$  bundle over  $\mathbb{P}^1$  is diagonalizable by Grothendieck's theorem ([8]). On the other hand we recall that on a Fano variety of Picard number one different from  $\mathbb{P}^1$ , a rank two vector bundle is decomposable if and only if its Grothendieck projectivization admits a section. A similar statement can be formulated in the case of flag bundles as follows.

PROPOSITION 3.6. *Let  $X$  be a smooth variety, and  $\pi : Y \rightarrow X$  be a  $G/B$ -bundle admitting a section  $\sigma : X \rightarrow Y$ . If  $H^1(X, \sigma^* K_j) = 0$  for all  $j \in D$ , then  $\pi$  is diagonalizable. In particular, if  $X$  is a Fano manifold of Picard number one,  $\pi$  is diagonalizable if and only if it admits a section.*

PROOF. Let  $\theta \in H^1(X, G)$  be the cocycle defining  $\pi$ . The existence of the section  $\sigma$  implies that the cocycle  $\theta$  lies in the image of the natural map  $H^1(X, B) \rightarrow H^1(X, G)$ , for some Borel subgroup  $B \subset G$ . Given any minimal parabolic  $P_i \supset B$ ,  $i \in D$ , we may consider the quotient of  $P_i$  by its unipotent radical and, subsequently, by its center, obtaining an epimorphism of groups:

$$P_i \rightarrow G_i \cong \mathrm{PGL}(2),$$

so that the image of  $B \subset P_i$  is a Borel subgroup  $B_i \subset G_i$ . The image  $\theta_i$  of  $\theta$  by the natural map  $H^1(X, B) \rightarrow H^1(X, B_i)$  defines then a  $\mathbb{P}^1$ -bundle  $\pi_i : Z_i \rightarrow X$ , together with a section  $\sigma_i$ , fitting in the diagram:

$$\begin{array}{ccc} Z_i & \xrightarrow{\quad} & Y \\ \sigma_i \uparrow \pi_i & \nearrow \sigma & \downarrow \rho_i \\ X & \xrightarrow{\rho_i \circ \sigma} & Y_i \end{array}$$

The existence of the section  $\sigma_i$  tells us that  $Z_i$  is the projectivization over  $X$  of a rank two vector bundle  $\mathcal{E}_i$ , fitting in an exact sequence:

$$0 \rightarrow \mathcal{O}_X(\sigma^* K_i) \rightarrow \mathcal{E}_i \rightarrow \mathcal{O}_X \rightarrow 0$$

This short exact sequence splits, by the assumptions, and we may conclude that the cocycle  $\theta_i \in H^1(X, B_i)$ , defining the  $\mathbb{P}^1$ -bundle  $Z_i$  over  $X$ , lies in the image of  $H^1(X, H_i)$ . Finally we take the map  $\psi : B \rightarrow \prod_{i \in D} B_i$  and consider the inverse image  $H$  of the subgroup  $\prod_i H_i$ , which is a Cartan subgroup of  $G$  contained in  $B$  (we may see this by looking at the corresponding subalgebras). It then follows that  $\theta \in H^1(X, H)$ , and so  $\pi : Y \rightarrow X$  is diagonalizable.

For the second part, assume that  $X$  is Fano manifold of Picard number one. If  $\pi$  admits a section then, in the case  $X \cong \mathbb{P}^1$  it is diagonalizable by Grothendieck's theorem while, if  $\dim(X) \geq 2$  the result follows by the first part of this statement and Kodaira vanishing. Conversely, if  $\pi$  is diagonalizable then its defining cocycle is in the image of the natural map  $H^1(X, H) \rightarrow H^1(X, G)$ , for some Cartan subgroup  $H$ . Then the cocycle is also in the image of the natural map  $H^1(X, B) \rightarrow H^1(X, G)$ , which in turn implies that the flag bundle has a section (see Section 2.1). ■

In other words, for flag bundles over a Fano manifold of Picard number one, reducibility and decomposability with respect to  $\emptyset$  are equivalent. Unfortunately, we cannot expect a similar result in the case of a general subset  $I \subsetneq D$ , as one can see in the following example.

**EXAMPLE 3.7.** Let  $Y$  be the complete flag over  $X = \mathbb{P}^3$ , and  $\pi$  be the natural projection. As a flag bundle, it is indecomposable but, considering  $I = \{2\}$ , so that  $Y_I \cong \mathbb{P}(T_{\mathbb{P}^3})$ , the projection  $\pi_I : Y_I \rightarrow X$  admits sections provided by any surjective morphism  $T_{\mathbb{P}^3} \rightarrow \mathcal{O}_{\mathbb{P}^3}(2)$ .

**3.1. Decomposability vs. reducibility.** Along this section  $\pi : Y \rightarrow X$  will denote a  $G/B$ -bundle that is reducible with respect to some  $I \subsetneq D$ , and  $\sigma_I : X \rightarrow Y_I$  the corresponding section. Given the associate parabolic subgroup  $P := P(I)$ , we may consider the cocycle  $\theta$  defining the bundle as an element in  $H^1(X, P)$ . We fix a Levi decomposition  $P \cong U \rtimes L$ , where  $U \triangleleft P$  is the unipotent radical of  $P$  and  $L \subset P$  is reductive.

We may now consider also the group  $G' = L/Z(L)$ , which is a semisimple linear algebraic group. The image of  $\theta$  into  $H^1(X, G')$  defines a flag bundle on  $X$ , that we denote here by  $\pi' : Y' \rightarrow X$ , fitting in the following diagram:

$$\begin{array}{ccccc}
 & & \pi & & \\
 & & \curvearrowright & & \\
 Y & \xrightarrow{\rho_I} & Y_I & \xrightarrow{\quad} & X \\
 \uparrow & & \uparrow \sigma_I & & \nearrow \\
 Y' & \xrightarrow{\pi'} & X & & 
 \end{array}$$

The first thing we may say is that, since a section of  $\pi'$  gives a section of  $\pi$ , a direct application of Proposition 3.6 tells us the following:

**PROPOSITION 3.8.** *Let  $X$  be a Fano manifold of Picard number one, and  $\pi : Y \rightarrow X$  be a flag bundle, reducible with respect to some subset  $I \subsetneq D$ . Then the  $G'/B'$ -bundle  $\pi'$  defined above is diagonalizable if and only if  $\pi$  is diagonalizable.*

More generally, one may write conditions for the decomposability with respect to  $I$ . Consider the Lie algebra  $\mathfrak{n}$  of  $U$ , which is the nilradical of the Lie algebra of  $P$ . The subgroup  $L \subset P$  acts on  $U$  by conjugation; we may consider  $L$  as the

quotient  $P/U$ , and its adjoint action on  $\mathfrak{n}$ , so that we may transport the cocycle  $\theta$  defining  $\pi$  via the induced maps

$$H^1(X, P) \longrightarrow H^1(X, L) \longrightarrow H^1(X, \text{Aut}(\mathfrak{n})),$$

sending  $(\theta_{ij})$  to  $(\text{Ad}_{\theta_{ij}^\ell})$ , where the superindex  $\ell$  indicates the Levi part of  $\theta$ , for the Levi decomposition fixed above (note that every  $\theta_{ij}$  can be decomposed as  $\theta_{ij} = \theta_{ij}^u \theta_{ij}^\ell$ , where  $\theta_{ij}^u(x) \in U$  and  $\theta_{ij}^\ell(x) \in L$ ). The result is precisely the cocycle defining the vector bundle  $\sigma_I^* \Omega_{Y_I|X}$ .

Note that, since  $U$  is unipotent, the exponential map from its Lie algebra  $\mathfrak{n}$  provides an homeomorphism (with the Zariski topology), and the inclusion  $U \subset P$  induces an homeomorphism of  $U$  and  $P/L$ , turning  $P$  into a  $L$ -principal bundle over  $\mathfrak{n}$ . Denoting by  $\psi$  the composition

$$P \longrightarrow P/L \longrightarrow U \xrightarrow{\log} \mathfrak{n},$$

which sends an element  $p \in P$  to the logarithm  $\log(p^u)$  of its unipotent part  $p^u \in U$ , and considering the cocycle  $\theta = (\theta_{ij}) \in H^1(X, P)$  (defined with respect to some covering  $\{\mathcal{U}_i\}$  of  $X$  in the analytic topology), we may define  $\xi_{ij} := \psi \circ \theta_{ij} := \mathcal{U}_{ij} \rightarrow \mathfrak{n}$ , and check, by writing explicitly the cocycle conditions, that these elements define a cocycle  $\xi \in H^1(X, \sigma_I^* \Omega_{Y_I|X})$ . In fact, taking the unipotent part of the cocycle conditions for  $\theta$  provides:

$$(\theta_{ij} \theta_{jk})^u = \theta_{ik}^u \Rightarrow \theta_{ij}^u (\theta_{ji}^\ell \theta_{jk}^u \theta_{ji}^\ell) = \theta_{ik}^u,$$

and this information defines a cocycle condition in  $H^1(X, \sigma_I^* \Omega_{Y_I|X})$ . Then, we may state the following:

**PROPOSITION 3.9.** *Let  $\pi : Y \rightarrow X$  be a flag bundle, reducible with respect to some subset  $I \subsetneq D$ . Then  $\pi$  is decomposable with respect to  $I$  if and only if the cocycle  $\xi \in H^1(X, \sigma_I^* \Omega_{Y_I|X})$  is zero.*

**PROOF.** Arguing as above, the bundle  $Y$  is decomposable with respect to  $I$  if  $\theta_{ij}^u(x) = 1$  for all  $x \in \mathcal{U}_{ij}$ , and all  $i, j$ . But this is equivalent to say that  $\xi_{ij}(x) = \psi(\theta_{ij}(x)) = 0$ .  $\blacksquare$

**REMARK 3.10.** This result can be seen as a generalization of the case of vector bundles, too. In fact, the obstruction for a vector bundle  $\mathcal{E}$  over  $X$  given as an extension

$$0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \rightarrow 0$$

to be a direct sum of  $\mathcal{E}'$  and  $\mathcal{E}''$  lies in  $H^1(X, \mathcal{E}' \otimes \mathcal{E}'')$ , which is precisely the  $H^1$  of the restriction of the relative cotangent bundle of the associated Grassmannian bundle to its section over  $X$  provided by the quotient  $\mathcal{E} \rightarrow \mathcal{E}''$ .

#### 4. Uniform flag bundles

Uniformity of flag bundles is an extension of a classical concept within the theory of vector bundles (cf. [20, §3]), that applies to a triple  $(X, \mathcal{M}, \mathcal{E})$ , where  $X$  is an algebraic variety,  $\mathcal{M}$  is a family of rational curves on  $X$ , and  $\mathcal{E}$  is a vector bundle on  $X$ . Then  $\mathcal{E}$  is said to be uniform with respect to  $\mathcal{M}$  if the (isomorphism class of the) pull-back of  $\mathcal{E}$  via the normalization of one of the curves of the family does not depend on the curve chosen.

Let us now consider a  $G/B$ -bundle  $\pi : Y \rightarrow X$  on a projective variety  $X$ , and a family of rational curves  $\mathcal{M}$  on  $X$ , with evaluation morphism  $q : \mathcal{U} \rightarrow X$ . We



may consider the pull-back  $q^*Y := Y \times_X \mathcal{U}$ , which is a  $G/B$ -bundle over  $\mathcal{U}$ , whose natural morphism onto  $\mathcal{U}$  will be denoted by  $\pi$ , by abuse of notation. Following [19, Section 3.3], for every rational curve  $\Gamma = p^{-1}(z) \subset \mathcal{U}$  the pull-back of the  $G/B$ -bundle  $q^*Y$  to  $\Gamma$  is determined by the Dynkin diagram of the group  $G$ , whose nodes are tagged by nonnegative integers  $d_i$ . Once the ordering of the nodes of the Dynkin diagram is fixed, the corresponding  $n$ -tuple  $\delta_\Gamma(Y) := (d_1, \dots, d_n)$  is called the *tag of the bundle at  $\Gamma$* . These integers may be interpreted as the degrees on a minimal section of  $q^*Y$  over  $\Gamma$  of the canonical divisors  $K_j$  of the elementary contractions  $\rho_j$  of  $Y$  extending the elementary contractions of  $G/B$  –see (1).

REMARK 4.1. In particular, in the case in which  $G/B$  is the complete flag manifold of a projective space, the tag of the bundle at a rational curve  $\Gamma$  can be easily computed from the splitting type of (a vector bundle defining) the corresponding projective bundle on  $\Gamma$ . If the splitting type of the bundle is  $(a_0, \dots, a_r)$ ,  $a_0 \leq \dots \leq a_r$ , the tagged Dynkin diagram is:

$$\begin{array}{ccccccc} a_1 - a_0 & & a_2 - a_1 & & a_3 - a_2 & & \cdots & & a_{r-1} - a_{r-2} & & a_r - a_{r-1} & & A_r \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ & \text{---} & \circ & \end{array}$$

Hence it makes sense to pose the following definition:

DEFINITION 4.2. Given a projective variety  $X$ , a dominating family of rational curves  $\mathcal{M}$  on  $X$ , and a flag bundle  $\pi : Y \rightarrow X$ , we say that  $Y$  is *uniform with respect to  $\mathcal{M}$*  if the tag  $\delta_\Gamma(Y)$  is independent of the choice of the curve  $\Gamma \in \mathcal{M}$ . In this case, the tag will be denoted by  $\delta(Y)$ , or simply by  $\delta$ .

REMARK 4.3. Note that, if  $\pi : Y \rightarrow X$  is a uniform flag bundle defined by a cocycle  $\theta \in H^1(X, G)$ , then every vector bundle determined by  $\theta$  and a given linear representation of  $G$  will be uniform in the classical sense.

EXAMPLE 4.4. Besides complete flag bundles defined by uniform projective bundles, the most obvious examples of uniform flag bundle are the products  $X \times G/B$ , that we call *trivial* flag bundles. Moreover, given a semisimple group  $G$ , and a maximal parabolic subgroup  $P \subset G$  corresponding to a simple root that is not exposed short ([13, Definition 2.10]), then the map  $G/B \rightarrow G/P$  is a  $G'/B'$ -bundle, where  $G'$  is a Levi part of  $P$  and  $B'$  is a Borel subgroup of  $B'$ , that is uniform with respect to the complete family of lines in  $G/P$ .

**4.1. Characterization of trivial flag bundles.** We will now consider the simplest case in which the uniformity of the flag bundle allows us to classify it. More concretely, we will study the case in which  $\delta(Y) = (0, \dots, 0)$ .

Throughout this section  $\pi : Y \rightarrow X$  will denote a flag bundle over a smooth projective manifold  $X$ . We will further assume that  $X$  is rationally chain connected with respect to  $s$  unsplit families of rational curves

$$\mathcal{M}_i \xleftarrow{p_i} \mathcal{U}_i \xrightarrow{q_i} X,$$

whose classes will be denoted by  $C_i$ ,  $i = 1, \dots, s$ . This means, by definition, that  $\mathcal{M}_i$  is a projective irreducible component of the scheme  $\text{RatCurves}^n(X)$ . We will show that the triviality of  $Y$  over the curves of the families  $\mathcal{M}_i$  implies the triviality of  $Y$  itself:

THEOREM 4.5. *Let  $X$  be a manifold which is rationally chain connected with respect to  $\mathcal{M}_1, \dots, \mathcal{M}_s$ , unsplit families of rational curves, and  $\pi : Y \rightarrow X$  a  $G/B$*

bundle over  $X$ . Assume that for every rational curve  $\Gamma_i := p_i^{-1}(z)$  we have  $\delta_{\Gamma_i}(Y) = (0, \dots, 0)$ . Then  $Y \cong X \times G/B$  is trivial as a  $G/B$ -bundle over  $X$ .

PROOF. For every  $i = 1, \dots, s$ , we pull back the  $G/B$ -bundle  $\pi : Y \rightarrow X$  to the universal family  $\mathcal{U}_i$  obtaining a  $G/B$ -bundle  $\overline{\mathcal{U}}_i$  over  $\mathcal{U}_i$ . By hypothesis, the composition with  $p_i$  defines  $\overline{\mathcal{U}}_i$  as a  $(\mathbb{P}^1 \times G/B)$ -bundle over  $\mathcal{M}_i$ , which is given by a cocycle in  $H^1(\mathcal{M}_i, G \times \text{Aut}(\mathbb{P}^1))$ , whose image under the natural map to  $H^1(\mathcal{M}_i, \text{Aut}(\mathbb{P}^1))$  defines the  $\mathbb{P}^1$ -bundle  $p_i : \mathcal{U}_i \rightarrow \mathcal{M}_i$ . On the other hand, its image into  $H^1(\mathcal{M}_i, G)$  defines a  $G/B$ -bundle  $\pi_i : \overline{\mathcal{M}}_i \rightarrow \mathcal{M}_i$ , and one can check that its pull-back via  $p_i$  to  $\mathcal{U}_i$  is  $\overline{\mathcal{U}}_i$ , so that we have a diagram with Cartesian squares, whose vertical arrows are  $G/B$ -bundles:

$$\begin{array}{ccccc} \overline{\mathcal{M}}_i & \xleftarrow{\overline{p}_i} & \overline{\mathcal{U}}_i & \xrightarrow{\overline{q}_i} & Y \\ \downarrow \pi_i & & \downarrow & & \downarrow \pi \\ \mathcal{M}_i & \xleftarrow{p_i} & \mathcal{U}_i & \xrightarrow{q_i} & X \end{array}$$

Moreover, the map  $\overline{q}_i$  defines a  $\mathbb{P}^1$ -bundle structure on  $\overline{\mathcal{U}}_i$ , so that we may consider it as a family of rational curves in  $Y$ , that may be identified with the family of minimal sections of  $Y$  over curves of the family  $\mathcal{M}_i$ . Note that the natural map  $\overline{\mathcal{M}}_i \rightarrow \text{RatCurves}^n(Y)$  is injective, and one can easily prove that its image is indeed an irreducible component of  $\text{RatCurves}^n(Y)$ .

The families of rational curves in  $Y$  parametrized by the  $\overline{\mathcal{M}}_i$ 's,  $i = 1, \dots, s$  define a *rational quotient* of  $Y$ , i.e. there exists a proper morphism  $\tau : Y^0 \rightarrow Z^0$ , defined on an open set  $Y^0 \subset Y$ , onto a normal variety  $Z^0$ , whose fibers are equivalence classes in  $Y^0$  of the relation defined by connectedness with respect to the families  $\overline{\mathcal{M}}_i$  (see [12, IV.4.16] for details).

A general fiber  $X'$  of  $\tau$  is a smooth projective variety which is rationally connected by the curves of the (unsplit) families  $\overline{\mathcal{M}}_i$  contained in it. This implies that the numerical class of every curve contained in  $X'$  is a linear combination of the numerical classes of the curves parametrized by the families  $\overline{\mathcal{M}}_i$ 's. In particular  $-K_j$  is numerically trivial on  $X'$  for every  $j \in D$ , hence trivial, being  $X'$  rationally connected. Therefore  $-K_\pi$ , which is an integral combination of the  $-K_j$ 's (see Lemma 2.1) is trivial on  $X'$ .

We claim now that the restriction of  $\pi$  to  $X'$  is necessarily finite onto  $X$ . The finiteness follows from the fact that  $X'$  cannot contain a curve contracted by  $\pi$ , since  $-K_\pi$  is  $\pi$  ample, while the surjectivity follows by the interpretation of each  $\overline{\mathcal{M}}_i$  as the family minimal sections over curves of  $\mathcal{M}_i$ , the triviality of  $Y$  on these curves, and the hypothesis on the rational chain connectedness of  $X$  with respect to them.

Now, adjunction tells us that

$$K_{X'} = (K_Y)_{|X'} = (K_\pi + \pi^* K_X)_{|X'} = (\pi^* K_X)_{|X'},$$

so  $\pi|_{X'}$  is an étale cover of  $X$ , contradicting that  $X$  is rationally chain connected, and hence simply connected, unless  $X'$  is a section of  $\pi$ .

This section satisfies the hypotheses of Proposition 3.6, so we may conclude that  $\pi : Y \rightarrow X$  is diagonalizable, i.e.,  $\pi$  is defined by a cocycle in  $H^1(X, (\mathbb{C}^*)^n) \simeq \text{Pic}(X)^n$ ; let  $L_1, \dots, L_n \in \text{Pic}(X)$  be the line bundles in  $X$  determined by this

cocycle. Since the restriction of  $Y$  to any rational curve of the families  $\mathcal{M}_i$  is trivial, it follows that  $L_1, \dots, L_n$  are trivial on each one of this curves. But  $X$  is rationally chain connected with respect to the families  $\mathcal{M}_i$ , therefore the line bundles  $L_j$  are numerically trivial. Finally, since  $X$  is simply connected, it follows that  $H^1(X, \mathcal{O}_X) = 0$ , and hence that the map  $\text{Pic}(X) \rightarrow H^2(X, \mathbb{Z})$  is injective: this tells us that the line bundles  $L_j$  are trivial, which is equivalent to say that the cocycle determining the bundle is trivial.  $\blacksquare$

As a consequence of Theorem 4.5, taking in account that a rational homogeneous bundle is trivial iff its associated flag bundle is trivial we get the following:

**COROLLARY 4.6.** *Let  $X$  and  $\mathcal{M}_1, \dots, \mathcal{M}_s$  be as in Theorem 4.5, and let  $\pi : E \rightarrow X$  be an  $F$ -bundle over  $X$ , with  $F$  rational homogeneous, satisfying that for the normalization  $f_i : \mathbb{P}^1 \rightarrow X$  of any curve of the family  $\mathcal{M}_i$ , and all  $i = 1, \dots, s$ , the fibre product  $\mathbb{P}^1 \times_X Y$  is trivial as an  $F$ -bundle over  $\mathbb{P}^1$ . Then  $E$  is trivial as an  $F$ -bundle over  $X$ .*

## 5. Diagonalizability criteria for uniform flag bundles

Along this section  $X$  will denote a Fano manifold of Picard number one and  $\pi : Y \rightarrow X$  a flag bundle, uniform with respect to an unsplit dominating family  $\mathcal{M}$  of rational curves, with tag  $\delta = (d_1, \dots, d_n)$ . The minimal sections of the bundle  $Y$  over the curves of the family and the compatibility among them—in order to construct sections of the  $Y_I$ 's over  $X$ —give rise to a concept of *uniform reducibility*, that we will discuss in Section 5.1. Then, in Section 5.2 we will study the differential of the morphism from  $\mathcal{U}$  to a certain  $Y_{I_0}$  (see below the definition of  $I_0$ ) to state some reducibility criteria for uniform flag bundles. In particular, we will show a flag bundle counterpart of the classic Grauert-Mülich theorem, together with some diagonalizability criteria for uniform bundles with special tagging.

**5.1. Uniform reducibility of uniform flag bundles.** With the same notation as above, let us denote

$$I_0 := \{i \in D \mid d_i = 0\}.$$

The Dynkin subdiagram of  $\mathcal{D}$  supported on  $I_0$  will be denoted by  $\mathcal{D}_{I_0}$  and  $P(I_0) \subset G$  will stand for the corresponding parabolic subgroup (so that the fibers of the submersion  $\rho_{I_0} : Y \rightarrow Y_{I_0}$  are flag manifolds associated to a semisimple subgroup of  $G$  determined by the Dynkin subdiagram  $\mathcal{D}_{I_0}$ ). In view of Theorem 4.5, we will always assume  $I_0 \subsetneq D$ . Then over every rational curve  $\Gamma$  of the family we have a well defined trivial subflag bundle  $F_{I_0} \times \Gamma \subset \pi^{-1}(\Gamma)$ , where  $F_{I_0}$  denotes a fiber of  $\rho_{I_0}$ . We may glue together this data to construct a morphism  $s_0 : \mathcal{U} \rightarrow Y_{I_0}$ . In fact, we may consider the family of minimal sections of  $Y$  over curves of the family  $\mathcal{M}$ , denoted by  $\bar{p} : \bar{\mathcal{U}} \rightarrow \bar{\mathcal{M}}$ . There is a commutative diagram:

$$\begin{array}{ccc} \bar{\mathcal{U}} & \xrightarrow{\bar{p}} & \bar{\mathcal{M}} \\ \downarrow & & \downarrow \\ \mathcal{U} & \xrightarrow{p} & \mathcal{M} \end{array}$$

whose vertical arrows are smooth morphisms with fibers isomorphic to  $F_{I_0}$ . We may consider the composition of the evaluation  $\bar{q} : \bar{\mathcal{U}} \rightarrow Y$  with  $\rho_{I_0}$ , that is constant on

the fibers of  $\overline{\mathcal{U}} \rightarrow \mathcal{U}$ , and so we obtain a map  $s_0 : \mathcal{U} \rightarrow Y_{I_0}$ , fitting in the following commutative diagram:

$$\begin{array}{ccccc}
 \overline{\mathcal{M}} & \xleftarrow{\overline{p}} & \overline{\mathcal{U}} & \xrightarrow{\overline{q}} & Y \\
 \downarrow & & \downarrow & \searrow & \downarrow \rho_{I_0} \\
 & & & & Y_{I_0} \\
 & & & \nearrow s_0 & \downarrow \pi_{I_0} \\
 \mathcal{M} & \xleftarrow{p} & \mathcal{U} & \xrightarrow{q} & X
 \end{array}$$

DEFINITION 5.1. With the same notation as above, given any set  $I \subsetneq D$  of nodes of  $\mathcal{D}$  containing  $I_0$ , we denote by  $\rho_{I_0, I} : Y_{I_0} \rightarrow Y_I$  the natural projection. We say that  $\pi : Y \rightarrow X$  is *uniformly reducible with respect to  $\mathcal{M}$  and  $I$* , or simply  $(\mathcal{M}, I)$ -reducible, if and only if the composition  $\rho_{I_0, I} \circ s_0$  factors via  $q : \mathcal{U} \rightarrow X$ , that is, if there exists a morphism  $\sigma_I : X \rightarrow Y_I$  such that the following diagram is commutative:

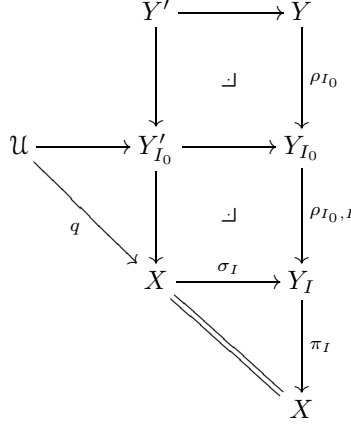
$$\begin{array}{ccc}
 \mathcal{U} & \xrightarrow{s_0} & Y_{I_0} \\
 q \downarrow & & \downarrow \rho_{I_0, I} \\
 X & \xrightarrow{\sigma_I} & Y_I
 \end{array}$$

Equivalently,  $Y$  is  $(\mathcal{M}, I)$ -reducible iff the cone  $s_{0*}(\text{NE}(\mathcal{U}|X))$  lies in the extremal face  $\text{NE}(Y_{I_0}|Y_I)$ . Then, given  $Y$ , either we may find a minimal subset  $I_0 \subseteq I \subsetneq D$  such that  $Y$  is  $(\mathcal{M}, I)$ -reducible, or  $Y$  is not  $(\mathcal{M}, I)$ -reducible with respect to  $I$ , for every  $I$ , and we say that  $\pi$  is *uniformly irreducible with respect to  $\mathcal{M}$* , or  $\mathcal{M}$ -irreducible, for short.

LEMMA 5.2. *If  $\pi$  is  $(\mathcal{M}, J_1)$ -reducible and  $(\mathcal{M}, J_2)$ -reducible, then it is also  $(\mathcal{M}, J_1 \cap J_2)$ -reducible.*

PROOF. Let us denote  $J = J_1 \cap J_2$ . By hypotheses, the maps  $\rho_{I_0, J_i} : \mathcal{U} \rightarrow Y_{J_i}$ ,  $i = 1, 2$ , factor via morphisms  $\sigma_{J_i} : X \rightarrow Y_{J_i}$ . For each  $i$  let us denote  $X'_i := \rho_{J, J_i}^{-1}(\sigma_{J_i}(X)) \subset Y_J$ . One can easily check that on every fiber  $\pi_J^{-1}(x)$ , the intersection  $X'_1 \cap X'_2$  is a point, then the map  $X'_1 \cap X'_2 \rightarrow X$  is a bijection and, since  $X$  is smooth, it is an isomorphism. Hence we have an inverse map  $\sigma_J : X \rightarrow Y_J$ , which satisfies  $\rho_{I_0, J} \circ s_0 = \sigma_J \circ q$ , by construction.  $\blacksquare$

In particular, when  $\pi$  is  $(\mathcal{M}, I)$ -reducible, the morphism  $s_0$  factors via the fiber product  $Y'_{I_0} := Y_{I_0} \times_{Y_I} X$ , which is a bundle over  $X$ , whose corresponding flag bundle is  $\pi' : Y' := Y \times_{Y_{I_0}} Y'_{I_0} = Y \times_{Y_I} X \rightarrow X$ . We then have a commutative diagram:



Within the problem of finding diagonalizability conditions for uniform flag bundles,  $(\mathcal{M}, I)$ -reducibility can be used to reduce the problem to flag bundles defined over groups of lower rank:

LEMMA 5.3. *With the same notation as above,  $Y'$  is a uniform flag bundle over  $X$ , whose tag  $\delta(Y') = (d'_i)_{i \in I}$  is a subtag of  $\delta(Y)$ , in the sense that  $d'_i = d_i$  for all  $I$  (considering  $I$  as a subset of  $D$ ). Moreover, if  $Y'$  is diagonalizable, then  $Y$  is diagonalizable.*

PROOF. The first part is immediate by construction. For the second note that, by Proposition 3.6, the diagonalizability of  $Y'$  is equivalent to the existence of a section of  $Y'$  over  $X$ , which in turn provides a section of  $Y$  over  $X$ . ■

REMARK 5.4. Note that, in many cases, the Dynkin diagram of the flag  $Y' \rightarrow X$  will be disconnected, and, according to Remark 3.5, the diagonalizability of  $Y'$  will be reduced to the diagonalizability of a certain number of uniform flag bundles over  $X$  associated to simple algebraic groups of smaller rank (one for each connected component of the Dynkin diagram of  $Y' \rightarrow X$ ).

As a consequence of Lemma 5.3, we have the following corollary:

COROLLARY 5.5. *With the same notation as above, if  $\pi : Y \rightarrow X$  is  $(\mathcal{M}, I_0)$ -reducible, then it is diagonalizable.*

PROOF. Arguing as above, we consider the uniform bundle  $Y' \rightarrow X$  whose tag is, in this case, equal to zero. We may then apply Theorem 4.5 to claim that  $Y'$  is trivial, hence it is diagonalizable and we may conclude by Lemma 5.3. ■

**5.2. Infinitesimal criteria for uniform reducibility.** Before starting, let us describe the set of hypotheses under which our results will work.

SETUP 5.6. As in the previous section, we consider here the case in which  $X$  is a Fano manifold of Picard number one and  $\pi : Y \rightarrow X$  is a flag bundle, uniform with respect to an unsplit dominating family of rational curves  $\mathcal{M}$ , that we will assume to be *complete*, in the sense that  $\mathcal{M}$  is an irreducible component of the scheme  $\text{RatCurves}^n(X)$  (cf. [12, II 2.11]). We will further assume that the evaluation morphism  $q : \mathcal{U} \rightarrow X$  is a quasi-elementary contraction, that is, it has connected fibers and the Picard number of  $\mathcal{U}$  is at most the sum of the Picard

numbers of  $X$  and of the general fiber of  $q$ . Finally, we will assume that  $\pi$  is not trivial, equivalently, with the notation of the previous section, that  $I_0 \subsetneq D$  (cf. Theorem 4.5).

REMARK 5.7. In practice, the technical assumptions on  $q$  are imposed in order to allow us to claim the following: any morphism  $s : \mathcal{U} \rightarrow Z$  satisfying that the restriction to a general fiber  $q^{-1}(x)$  is constant, factors via  $q : \mathcal{U} \rightarrow X$ . The property is fulfilled for lines in homogeneous manifolds of Picard number one; to our best knowledge, studying which families of rational curves satisfy this property is an open problem.

Let us now consider the morphism  $s_0 : \mathcal{U} \rightarrow Y_{I_0}$  defined in Section 5.1, and the composition  $\rho_{I_0, I} \circ s_0 : \mathcal{U} \rightarrow Y_I$ , for  $I \subset D$  a subset containing  $I_0$ . By the previous remark, the  $(\mathcal{M}, I)$ -reducibility of  $Y$ , reduces to the constancy of the map  $\rho_{I_0, I} \circ s_0$  on the general fiber  $q^{-1}(x)$ , a condition that we may write in terms of the differential of  $s_0$ , as follows:

LEMMA 5.8. *With the same notation as above,  $\pi : Y \rightarrow X$  is  $(\mathcal{M}, I)$ -reducible if and only if, at the general smooth point  $x$  of  $\mathcal{U}$  we have an inclusion  $ds_0(T_{\mathcal{U}|X, x}) \subset (s_0^*T_{Y_{I_0}|Y_I})_x$ , i.e. if the composition*

$$T_{\mathcal{U}|X, x} \xrightarrow{ds_0} (s_0^*T_{Y_{I_0}|X})_x \longrightarrow (s_0^*\rho_{I_0, I}^*T_{Y_I|X})_x$$

*is zero.*

In many cases, the particular geometry of the fibers of  $q$  allow us to improve the above criterion. Before going into details, let us introduce the following definition.

DEFINITION 5.9. Given an irreducible complex projective variety  $M$ , we define its *contractibility dimension*, denoted  $\text{cdim}(M)$ , as the maximum integer  $r$  satisfying that every morphism  $f : M \rightarrow M'$  whose image has dimension smaller than  $r$  is constant. Given a dominant projective morphism between irreducible varieties  $g : M \rightarrow N$ , we denote its dimension and its contractibility dimension by  $\dim(g) := \dim(g^{-1}(x))$ ,  $\text{cdim}(g) := \text{cdim}(g^{-1}(x))$ , for general  $x \in g(M)$ .

EXAMPLE 5.10. We will later consider the contractibility dimension of the evaluation  $q : \mathcal{U} \rightarrow X$ , that can be easily computed in many interesting examples, such as the family of lines on a rational homogeneous manifold of Picard number one. For instance, in the case in which the fibers of  $q$  are homogeneous manifolds of the form  $G/P$ , the contractibility dimension of  $q$  is the minimum of the dimensions of the manifolds  $G/P'$ , where  $P' \supset P$  is a parabolic subgroup containing  $P$ .

REMARK 5.11. If  $g : M \rightarrow N$  is a quasi-elementary contraction between normal projective varieties, and  $g' : M' \rightarrow N$  is a surjective projective morphism satisfying that  $\dim(g') < \text{cdim}(g)$ , then any morphism  $f : M \rightarrow M'$  satisfying  $g' \circ f = g$  factors via  $g$ , that is, there exists a morphism  $\sigma : N \rightarrow M'$  such that  $\sigma \circ g = f$ . In particular,  $\sigma$  is a section of  $g'$ :

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ & \searrow g & \nearrow \sigma \\ & N & \nwarrow g' \end{array}$$

We will apply the above ideas to the case of  $q : \mathcal{U} \rightarrow X$ . To begin with, we may state the following straightforward result:

LEMMA 5.12. *With the same notation as above,  $\pi : Y \rightarrow X$  is  $(\mathcal{M}, I)$ -reducible if and only if, at the general smooth point  $x$  of  $\mathcal{U}$  the composition*

$$(2) \quad T_{\mathcal{U}|X,x} \xrightarrow{ds_0} \left( s_0^* T_{Y_{I_0}|X} \right)_x \longrightarrow \left( s_0^* \rho_{I_0,I}^* T_{Y_I|X} \right)_x$$

has rank smaller than  $\text{cdim}(q)$ .

In the spirit of [5, Proposition 3.2], rather than looking at the map (2) at general points of a fiber  $q^{-1}(x)$ , we will look at its behaviour along a general fiber of  $\mathcal{U}$  over  $\mathcal{M}$ , obtaining conditions on the tag of a uniform bundle for its reducibility or diagonalizability. More concretely, let  $\rho_{I_0} : s_0^* Y = Y \times_{Y_{I_0}} \mathcal{U} \rightarrow \mathcal{U}$  be the pull-back bundle, fitting in the diagram:

$$\begin{array}{ccc} s_0^* Y & \xrightarrow{s_0} & Y \\ \rho_{I_0} \downarrow & & \downarrow \rho_{I_0} \\ \mathcal{U} & \xrightarrow{s_0} & Y_{I_0} \end{array}$$

Let  $\Gamma$  be a general fiber of  $\mathcal{U}$  over  $\mathcal{M}$ , and  $\bar{\Gamma}$  be any minimal section of  $\rho_{I_0}$  over  $\Gamma$  (note that, by Lemma 5.3,  $\Gamma \times_{Y_{I_0}} Y$  is trivial). Let us study the pull-back map:

$$(3) \quad (\rho_{I_0}^* T_{\mathcal{U}|X})_{|\bar{\Gamma}} \xrightarrow{\rho_{I_0}^* ds_0} \left( s_0^* \rho_{I_0}^* T_{Y_{I_0}|X} \right)_{|\bar{\Gamma}} \longrightarrow \left( s_0^* \rho_I^* T_{Y_I|X} \right)_{|\bar{\Gamma}}.$$

The completeness of the family  $\mathcal{M}$  allows us to claim that

$$(\rho_{I_0}^* T_{\mathcal{U}|X})_{|\bar{\Gamma}} \cong \mathcal{O}_{\bar{\Gamma}}(-1)^{\oplus \dim(q)}.$$

This in fact follows by the standard description of the differential morphism of the evaluation  $q : \mathcal{U} \rightarrow X$  (cf. [12, II 3.4]).

The splitting type of the target of (3) may be controlled by taking an admissible ordering  $\{L_1, \dots, L_m\}$  of  $\Phi$  compatible with  $I$ , which provides a filtration (see Section 2.2):

$$0 = \bar{\mathcal{E}}_{m-k} \subset \bar{\mathcal{E}}_{m-k-1} \subset \dots \subset \bar{\mathcal{E}}_0 = \rho_I^* T_{Y_I|X}$$

with quotients:  $\bar{\mathcal{E}}_r / \bar{\mathcal{E}}_{r+1} \simeq L_{m-r} \in \Phi^+ \setminus \Phi^+(I)$ , for all  $r$ . Summing up we get:

PROPOSITION 5.13. *Assume that the evaluation morphism  $q : \mathcal{U} \rightarrow X$  has contractibility dimension  $m$ , and that*

$$\#\{L_i \in \Phi^+ \setminus \Phi^+(I) \mid L_i \cdot \bar{\Gamma} \leq -1\} < m.$$

*Then  $\pi : Y \rightarrow X$  is  $(\mathcal{M}, I)$ -reducible.*

As a first application of Proposition 5.13 we obtain a flag bundle counterpart of the standard Grauert-Mülich theorem for vector bundles, that may be used, together with Lemma 5.3 and Remark 5.4, in the problem of diagonalizability of low rank uniform flag bundles on Fano manifolds.

PROPOSITION 5.14. *If  $I_1 := \{i \in D \mid d_i \leq 1\}$  is a proper subset of  $D$ , then  $\pi : Y \rightarrow X$  is  $(\mathcal{M}, I_1)$ -reducible.*

PROOF. Since by hypothesis we have that  $L_i \cdot \bar{\Gamma} \leq -2$  for all  $L_i \in \Phi^+ \setminus \Phi^+(I_1)$ , we conclude by Proposition 5.13.  $\blacksquare$

We will now state the main result of this section, for which we need to introduce some notation. For every index  $j \in I_1 \setminus I_0$ , that is, such that  $d_j = 1$ , we define by  $\mathcal{D}'_0(j)$  be the Dynkin subdiagram of  $\mathcal{D}$  supported on  $I_0 \cup \{j\}$ , and by  $\mathcal{D}_0(j)$  the connected component of  $\mathcal{D}'_0(j)$  containing the node  $j$ . We denote by  $I_0(j)$  the set of indices of  $\mathcal{D}_0(j)$ , and by  $m_j$  the number of positive roots  $L$  of  $G$  of the form:

$$L = - \sum_{r \in I_0(j)} a_r K_r, \quad a_r \geq 0, \quad a_j = 1.$$

For the reader's convenience, we include here the values  $m_j$  for every possible  $j$ , and every possible connected Dynkin diagram  $\mathcal{D}_0(j)$ :

$\mathcal{D}_0(j)$	$j$	$m_j$
$A_n$	$j$	$j(n+1-j)$
$B_n$	$j$	$j(2n-2j+1)$
$C_n$	$(j < n, n)$	$\left( j(2n-2j), \frac{n(n+1)}{2} \right)$
$D_n$	$(j < n-2, n-2, n-1, n)$	$\left( j(2n-2j), 4(n-2), \frac{n(n-1)}{2}, \frac{n(n-1)}{2} \right)$
$E_6$	$(1, 2, 3, 4, 5, 6)$	$(16, 20, 20, 18, 20, 16)$
$E_7$	$(1, 2, 3, 4, 5, 6, 7)$	$(32, 35, 30, 24, 30, 32, 27)$
$E_8$	$(1, 2, 3, 4, 5, 6, 7, 8)$	$(64, 56, 42, 30, 40, 48, 54, 56)$
$F_4$	$(1, 2, 3, 4)$	$(14, 12, 6, 8)$
$G_2$	$(1, 2)$	$(2, 4)$

TABLE 1. Values of  $m_j$  for connected Dynkin diagrams

THEOREM 5.15. *Let  $X$  be a Fano manifold,  $\mathcal{M}$  be an unsplit dominating complete family of rational curves, whose evaluation morphism  $q : \mathcal{U} \rightarrow X$  is a quasi-elementary contraction. Let  $\pi : Y \rightarrow X$  uniform  $G/B$ -bundle over  $X$ , with tag  $(d_1, \dots, d_n)$ , and consider, for every node  $j \in I_1 \setminus I_0$ , the integer  $m_j$  defined above. If  $\text{cdim}(q) > m_j$ , for every  $j \in I_1 \setminus I_0$ , then  $\pi$  is diagonalizable.*

PROOF. We will show that  $\pi$  is  $(\mathcal{M}, D \setminus \{j\})$ -reducible for every  $j \in I_1 \setminus I_0$ . Since  $\pi$  is also  $(\mathcal{M}, I_1)$ -reducible (Proposition 5.14), it follows by Lemma 5.2 that  $\pi$  is  $(\mathcal{M}, I_0)$ -reducible, hence diagonalizable by Corollary 5.5.

Fix an index  $j \in I_1 \setminus I_0$ , and denote  $J := D \setminus \{j\}$ . Take an admissible ordering of  $\Phi$  compatible with  $I_0 \subsetneq J$  (see Definition 2.2), and the corresponding filtration of  $\rho_J^* T_{Y_J|X}$ , whose quotients are isomorphic to classes  $L_{m-r} \in \Phi^+ \setminus \Phi^+(J)$ . Note that these are precisely the positive roots of  $G$  containing  $-K_j$  as a summand. All these classes have negative intersection with the minimal section  $\bar{\Gamma}$ , and, in order to apply Proposition 5.13, we need to count those for which  $L_{m-r} \cdot \bar{\Gamma}$  is equal to  $-1$ . This occurs only if  $L_{m-r}$  belongs to the root subsystem determined by the Dynkin subdiagram  $\mathcal{D}'_0(j)$ . Since this is the disjoint union of the root systems determined by the connected components of  $\mathcal{D}'_0(j)$ , one such  $L_{m-r}$  is necessarily a positive root for the connected Dynkin subdiagram  $\mathcal{D}_0(j)$ , containing  $-K_j$  as a summand with



multiplicity one (being  $-K_j \cdot \bar{\Gamma} = -1$ ). As there are  $m_j < \text{cdim}(q)$  of these classes  $L_{m-r}$ , we conclude that  $\pi$  is  $(\mathcal{M}, J)$ -reducible by Proposition 5.13.  $\blacksquare$

As a straightforward corollary, we remark that in the case  $\text{cdim}(q) \geq 2$ , diagonalizability follows from the positivity of the tag. Note that the condition  $\text{cdim}(q) \geq 2$  is obviously necessary, since the flag bundle determined by the universal bundle on any Grassmannian of lines is not diagonalizable, although it has tag equal to (1).

**COROLLARY 5.16.** *Let  $X$  be a Fano manifold,  $\mathcal{M}$  be an unsplit dominating family of rational curves, with evaluation morphism  $q$  satisfying that  $q^{-1}(x)$  does not admit non constant morphisms to a curve, for general  $x \in X$ . Let  $\pi : Y \rightarrow X$  be a uniform  $G/B$ -bundle over  $X$ . Then  $\pi$  is diagonalizable unless  $I_0 \neq \emptyset$ , that is, unless its tag contains a zero.*

Applied to uniform vector bundles, Corollary 5.16 provides the following statement, that, to the best of our knowledge, is new even for  $X \cong \mathbb{P}^n$ ,  $n \geq 3$ :

**COROLLARY 5.17.** *Let  $X$  be a Fano manifold,  $\mathcal{M}$  be an unsplit dominating family of rational curves, with evaluation morphism  $q$  satisfying that  $q^{-1}(x)$  does not admit non constant morphisms to a curve, for general  $x \in X$ . Let  $\mathcal{E}$  be a vector bundle over  $X$ , uniform with respect to  $\mathcal{M}$ , with splitting type  $(a_1, \dots, a_r)$ ,  $a_1 < \dots < a_r$ . Then  $\mathcal{E}$  is a direct sum of line bundles.*

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DEPARTAMENTO DE MATEMÁTICA APLICADA, ESCET, UNIVERSIDAD REY JUAN CARLOS,  
28933-MÓSTOLES, MADRID, SPAIN

*E-mail address:* roberto.munoz@urjc.es

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TRENTO, VIA SOMMARIVE 14 I-38123 POVO  
DI TRENTO (TN), ITALY

*E-mail address:* gianluca.occhetta@unitn.it

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TRENTO, VIA SOMMARIVE 14 I-38123 POVO  
DI TRENTO (TN), ITALY

*E-mail address:* lesolac@gmail.com